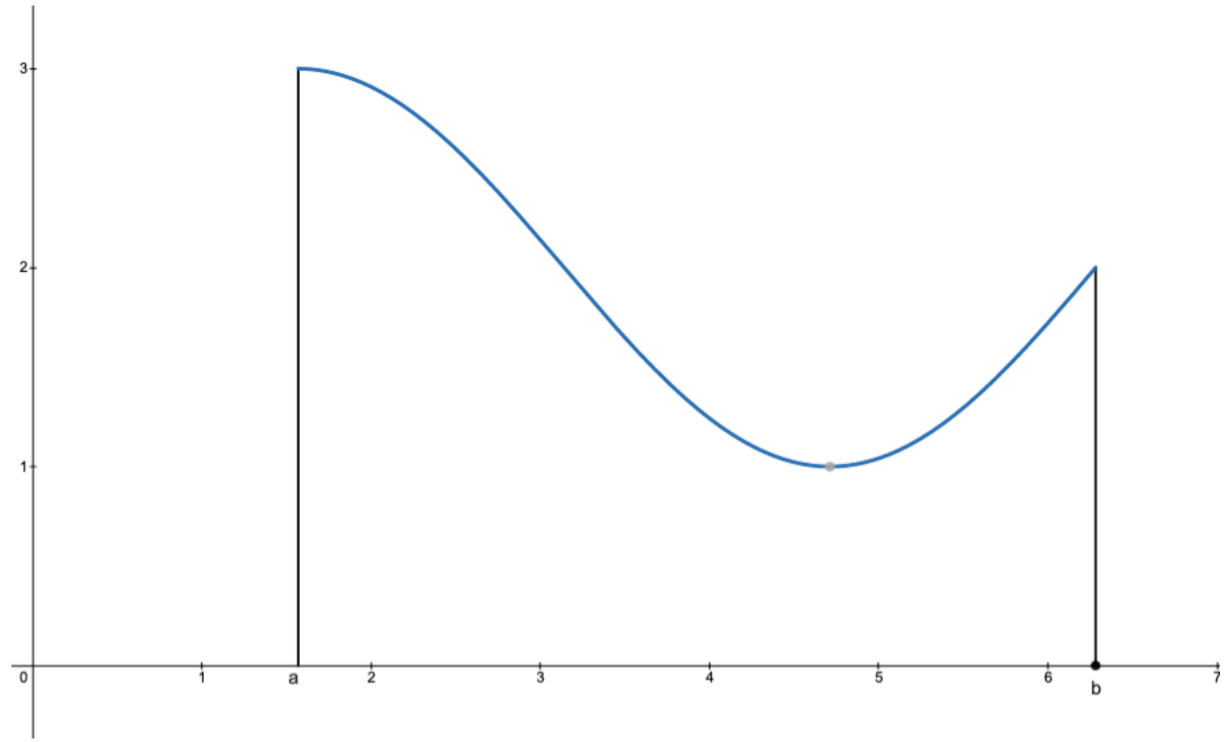
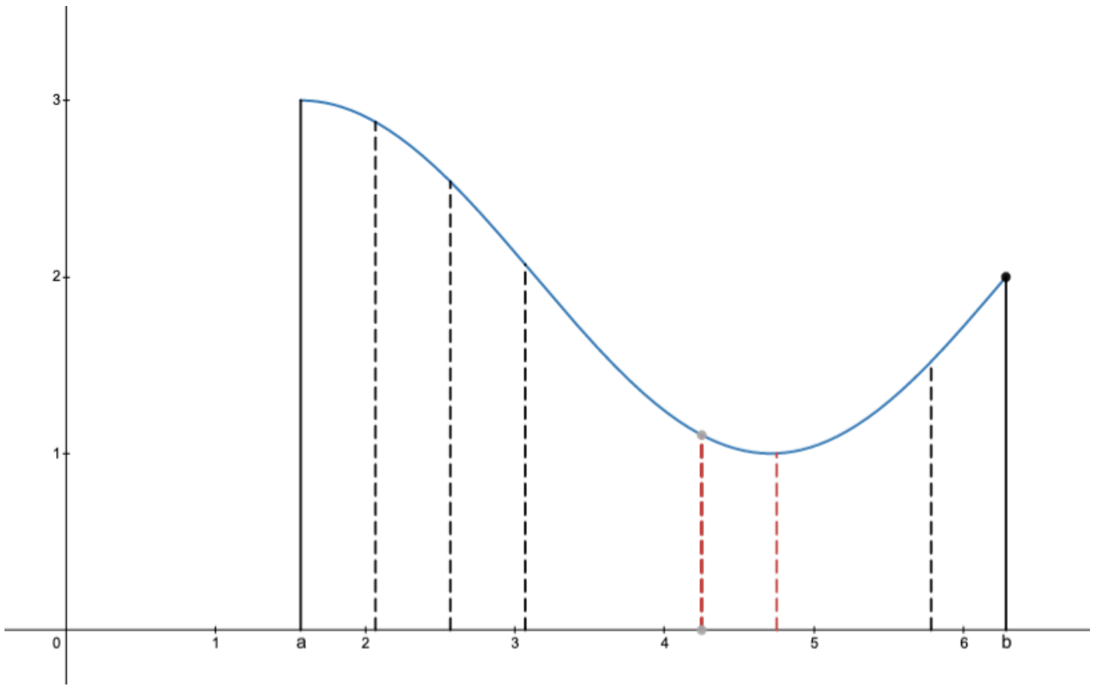
8.1 Arc Length

Suppose we are given a continuous, smooth ($f' \neq 0$) function f(x) defined over [a,b]. How might we find the length?



See Desmos animation on 5B page: https://www.desmos.com/calculator/jo1tkthwmq

Derivation of arc length formula: $y = f(x); a \le x \le b$



$$L = \int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx = \int_c^d \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Example: Find the length of the curve given by $f(x)=\frac{x^2}{2}-\frac{\ln x}{4}; \quad 2\le x\le 4$

Similarly, if $x = g(y); c \le y \le d$, we can show. $L = \int_c^d \sqrt{1+\left(g'(y)\right)^2}\,dy = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

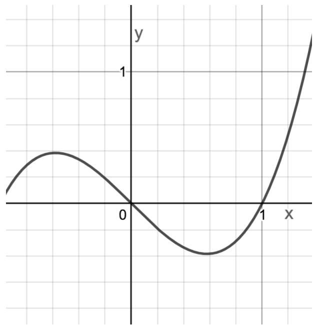
Example: Find the length of the curve given by $y^3 = x^2$ from (0,0) to (8,4).

Can be done either way, which is easier?

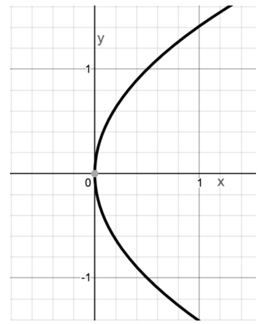
10.1: Parametric Equations in R^2

(For an interesting introduction to parametric equations, see "Quick Intuition about Parametric Equation"s on the 5B page)

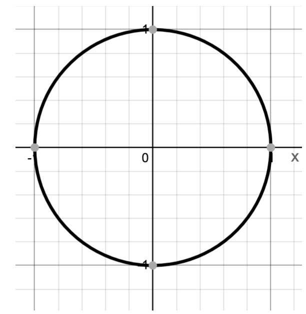
We have learned that the graphs of equations in two variables are curves in R^2 showing a relationship between the two variables, suppose x and y. Sometimes the relation can be expressed as a function, other times it cannot.



$y = f(x) = x^3 - x$



$x = g(y) = y^2$



$x^2 + y^2 = 1$

In this section, we will consider another way to write equations for curves in R^2, called__________________________.
This gives us the opportunity to

(1) ____ and

(2) ____(later)

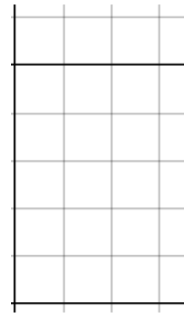
Showing dependence on a third variable

ice cream = $(\text{sunscreen})^2$

Often, the parameter is *time*.

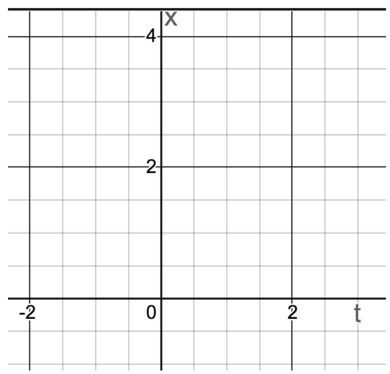
Example: Sketch $\begin{cases} x = 1 - t \\ y = t^2 \end{cases}$

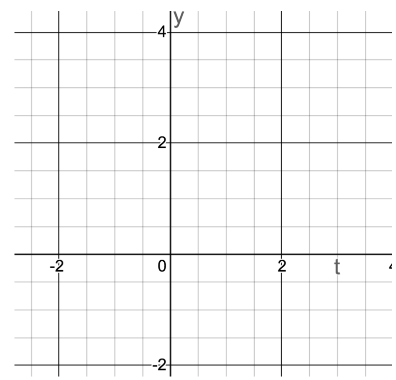
Methods:



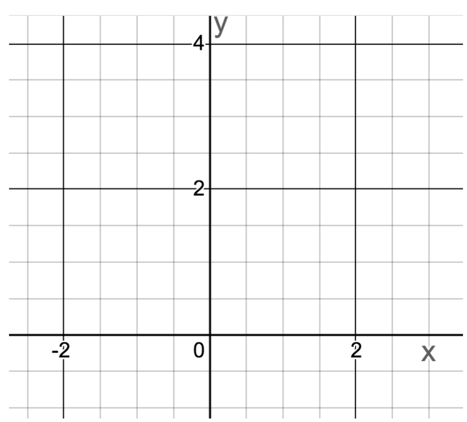
1. Plot points (last resort!)

2. Consider the graphs x(t) and y(t) separately to determine horizontal and vertical behavior. Then combine these “behaviors”.





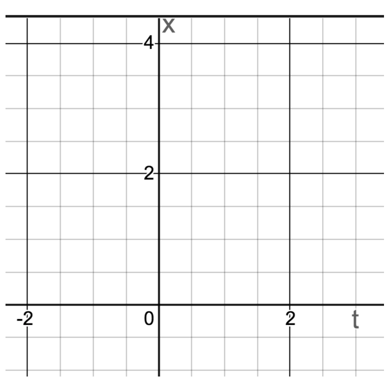
3. Eliminate the parameter (best, if possible). Caution: ____

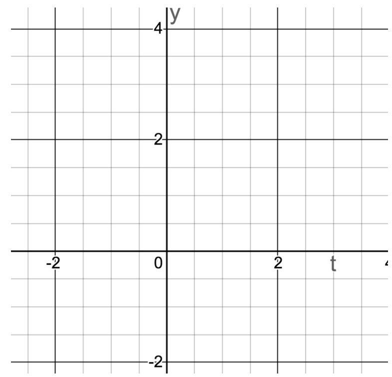


What if the problem was $\begin{cases} x = 1 - t^2 \\ y = t^4 \end{cases}$

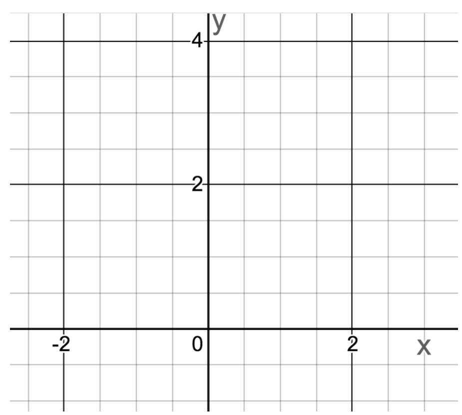
Always show direction of increasing t.

What if the problem was $\begin{cases} x = 1 - t^2 \\ y = t^4 \end{cases}$

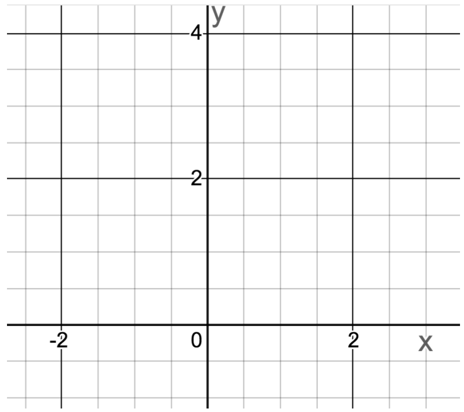




1. Eliminate the parameter (best, if possible). Caution: ______________________________



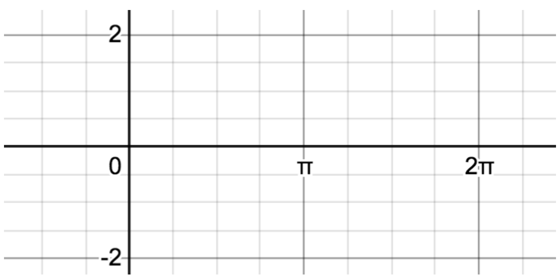
Example: Sketch $\begin{cases} x = e^t \\ y = e^{2t} \end{cases}$

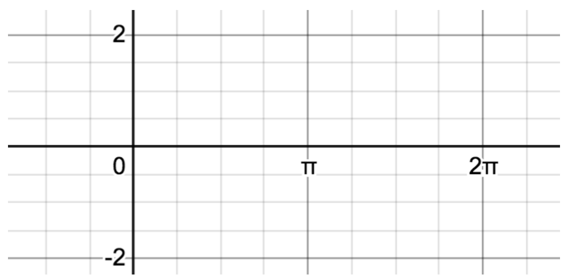


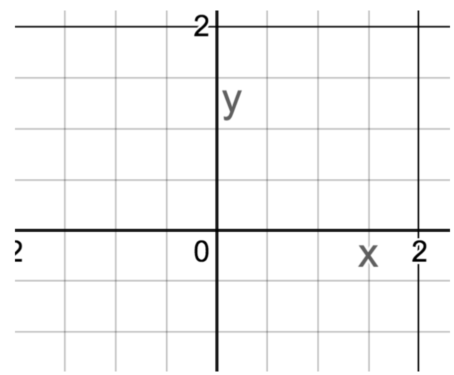
Expressing an equation that is not a function in terms of two which are.

Example: Sketch $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$

First consider x(t) and y(t) separately



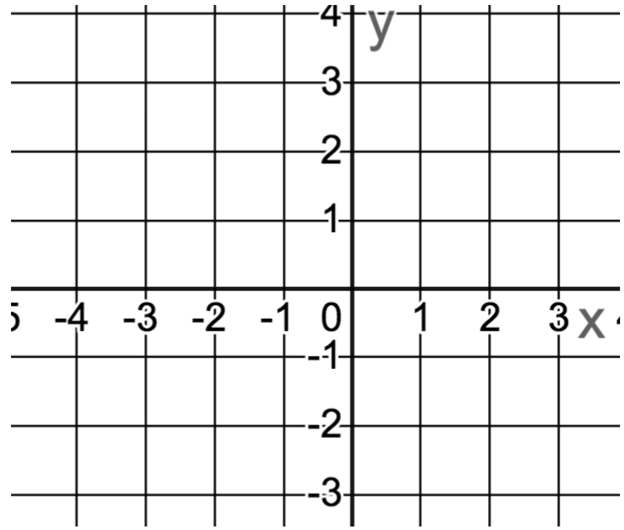




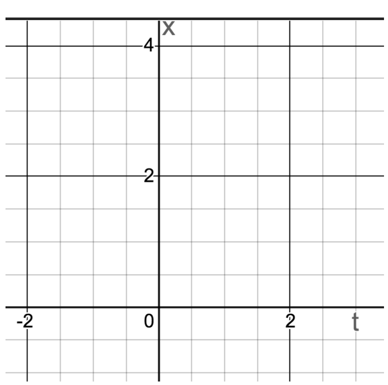
animation from 5B page: https://www.desmos.com/calculator/vslkzgeocx

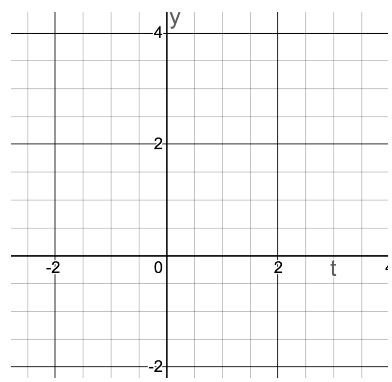
Example: Direction of increasing t

Sketch $\begin{cases} x = 2\sin t \\ y = 3\cos t \end{cases}$



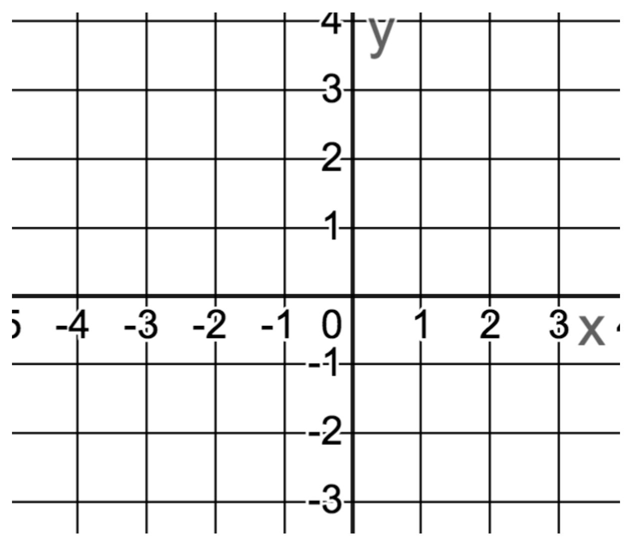
Consider x(t), y(t) if needed





Sometimes, only a portion of the curve is specified

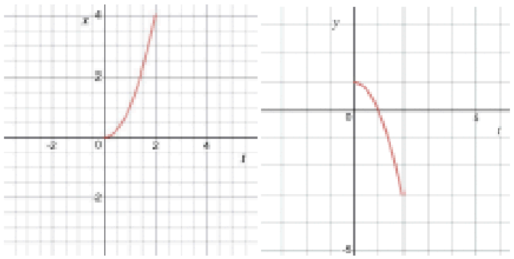
Example: Sketch $\begin{cases} x = 2\sin t \\ y = 3\cos t \end{cases}$ $\quad \frac{\pi}{2} \le t \le \pi$

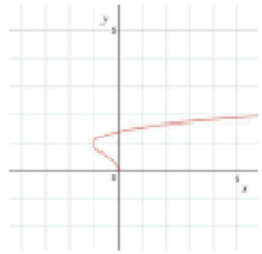


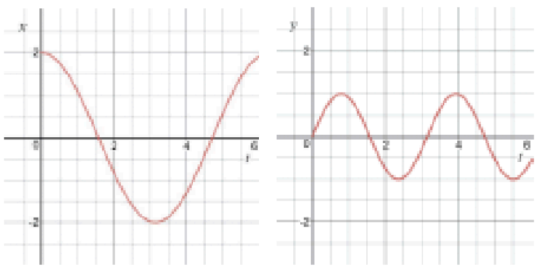
More interesting animation from 5B page: https://www.desmos.com/calculator/w9ab0dxrp3

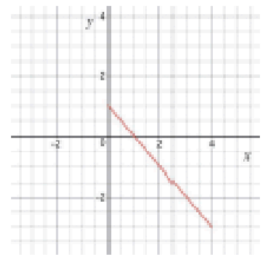
Additional Problems on Parametric Equations

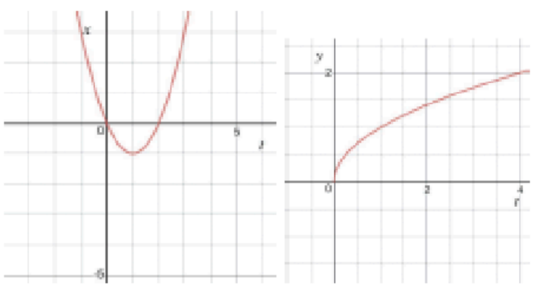
Match the graphs of the parametric pair x(t) and y(t) on the left with the graph in the xy plane on the right.

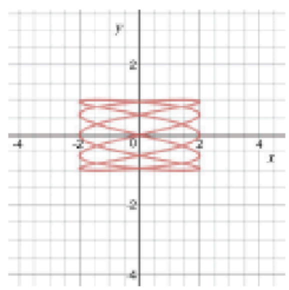


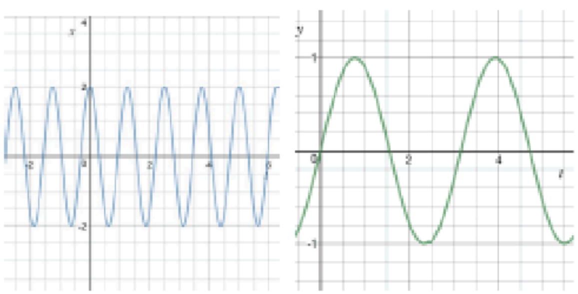


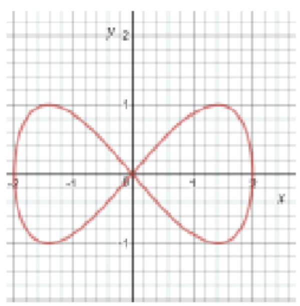








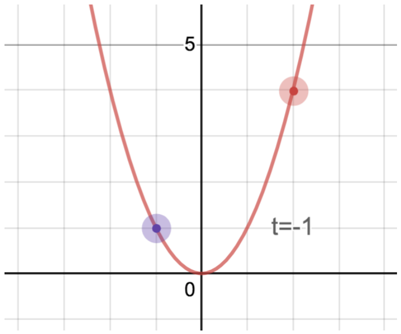




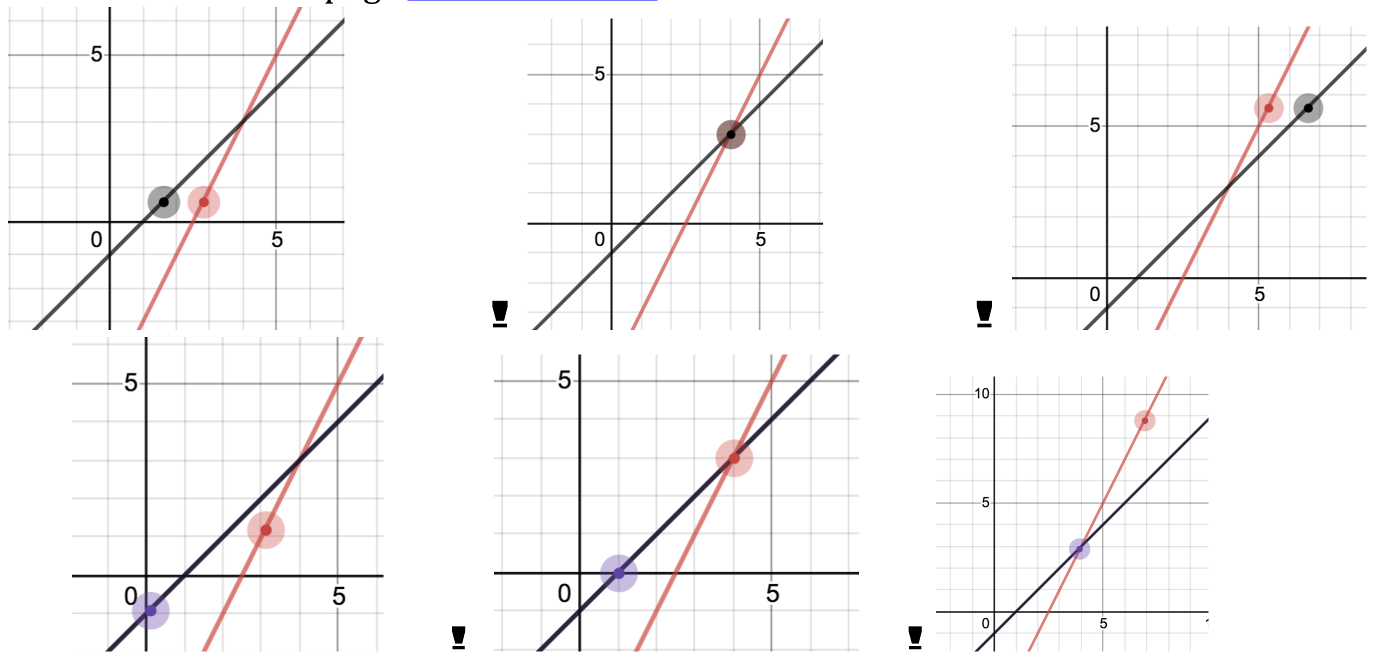
Parameterization is not unique: [5B page: parameterization is not unique]

$$\begin{cases} x = t \\ y = t^2 \end{cases}$$

$$\begin{cases} x = 1 - t \\ y = (1 - t) \end{cases}$$



Cross paths vs collision: 5B page Collide vs Cross



10.2 Calculus of Parametric Equations (derivatives, tangents and length)

Given $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$, if we can eliminate the parameter we get ______________________

To find $F'(x) = \frac{dy}{dx}$, chain rule: $F'(x) = \frac{dy}{dx} =$____, for f, g differentiable and $\frac{dx}{dt} \neq 0$

Example: Given $\begin{cases} x = e^t \\ y = e^{2t} - 1 \end{cases}$, if we express these equations in the form $y = F(x)$ find $F'(x) = \frac{dy}{dx}$ both directly, and by eliminating the parameter.

Example: For $\begin{cases} x = 2t^3 \\ y = 1 + 4t - t^2 \end{cases}$,

(a) Find an equation of the tangent line at (2,4)

(b) At what point on the above curve is the slope of the tangent line 1?

(c) Find $\dfrac{d^2y}{dx^2}$

Arclength: For a function $y = F(x)$ which can be expressed parametrically as $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$

$$L = \int_a^b \sqrt{1 + \left(F'(x)\right)^2}\, dx = \int_c^d \sqrt{1 + \left(\frac{\quad}{\quad}\right)^2}\, dx = \int_c^d \sqrt{\left(\frac{\quad}{\quad}\right)^2 + \left(\frac{\quad}{\quad}\right)^2}\, dx =$$

Example: Find the length of the curve given by $\begin{cases} x = e^t + e^{-t} \\ y = 5 - 2t \\ 0 \le t \le 3 \end{cases}$